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A conformal Petrov–Galerkin method for convection-dominated problems

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SUMMARY

In this paper, we present the 'conformal Petrov–Galerkin' (CPG) method in order to solve the 2D convection–diffusion equation on meshes composed of triangular elements. By 'conformal' it is meant that the discrete system is obtained from the continuous weak formulation by appropriately selecting different finite-dimensional subspaces for the shape and test functions without any additional stabilizing term. Our approach is based on searching continuous test functions that provide exact nodal values for a selected class of solutions. This method induces a stabilizing upwinding effect that removes the wiggles obtained with the Galerkin method. Copyright © 2008 John Wiley & Sons, Ltd.

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KEY WORDS: convection–diffusion; Petrov–Galerkin method; finite element method; numerical stabilization; streamline upwinding; boundary layers

1. INTRODUCTION

The Galerkin finite element (FE) method is very well suited for diffusion-dominated problems while it performs quite badly when transport effects prevail. Therefore, the solution of the convection–diffusion equation has been the object of extensive investigations, since this generic equation exhibits the principal numerical difficulties to be addressed for the solution of convection-dominated problems. A single parameter, the Péclet number, *Pe* (the ratio of transport *versus* diffusion effects), governs this equation. At low *Pe*, when diffusion dominates, the system is close to a functional minimization problem [1]. Accordingly, the solution wiggles tend to be strongly reduced (in the 'energy norm'). On the contrary, at high *Pe*, when transport prevails, the discrete eigenvalues tend to become purely imaginary and the discrete system no longer exhibits any natural wiggle reduction effect. To solve the problem, Brooks and Hughes developed the streamline upwind/Petrov–Galerkin

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(SUPG) technique in order to remove the solution oscillations at high *Pe* [2–6]. The SUPG method is consistent since the exact solution is a solution of the discrete problem. However, it is not a conformal Petrov–Galerkin method (CPG) [7, 8] in the sense that the discrete system is not exactly obtained by introducing appropriate finite-dimensional shape and test function spaces into the original continuous weak formulation. Nevertheless, defining a consistent and conformal FE method (such as Galerkin's method) in order to solve the convection–diffusion problem represents a highly attractive objective (which is the aim of this paper) and it should be emphasized that this elegant approach was never completely investigated, except in Perella's pioneer work [9].

It is worth noting that the present work is in keeping with the general pattern of investigations on the numerical solution of the convection–diffusion equation, from which very attractive alternative techniques, such as the residual-free bubbles FE method [10, 11] and the discontinuous Galerkin method [12, 13], have been proposed by various authors.

2. PRINCIPLES OF THE METHOD

2.1. Strong formulation

Let Ω and Γ denote an open-bounded set of $\mathbb{R}^2$ and its boundary. The strong normalized boundary value problem consists in searching a 'temperature' field $T(\mathbf{x})$ such that

$$\mathbf{v}\cdot\nabla T-\frac{1}{Pe}\Delta T=0 \quad \text{in } \Omega \tag{1}$$

$$T=\overline{T} \quad \text{on } \Gamma \tag{2}$$

where *Pe* and $\mathbf{v}$ stand for the global Péclet number and an assumed divergence-free vector field, while $\overline{T}:\Gamma\to\mathbb{R}$ is a prescribed function.

2.2. Weak formulation

The weak formulation of the boundary value problem (1)–(2) is expressed as follows:

Find $T\in S$ such that

$$\forall T'\in V, \quad \int_\Omega T'\mathbf{v}\cdot\nabla T\,\mathrm{d}\Omega+\frac{1}{Pe}\int_\Omega \nabla T'\cdot\nabla T\,\mathrm{d}\Omega=0 \tag{3}$$

with the function spaces

$$S=\{T\in H^1(\Omega)|T=\overline{T} \text{ on } \Gamma\} \quad \text{and} \quad V=\{T\in H^1(\Omega)|T=0 \text{ on } \Gamma\}=H_0^1(\Omega)$$

2.3. Galerkin FE approximation

The discrete Galerkin FE problem corresponding to the previous weak formulation is expressed as follows:

Find $T^h\in S^h$ such that

$$\forall T^{h\prime}\in V^h, \quad \int_\Omega T^{h\prime}\mathbf{v}\cdot\nabla T^h\,\mathrm{d}\Omega+\frac{1}{Pe}\int_\Omega \nabla T^{h\prime}\cdot\nabla T^h\,\mathrm{d}\Omega=0 \tag{4}$$

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where the FE subspaces $S^h \subset S$ and $V^h \subset V$ consist of continuous piecewise linear polynomials defined on a conforming triangulation $\mathscr{T}^h$ of Ω.

The functions of S^h and V^h can be expressed as linear combinations of the Lagrange shape functions ϕ_i. For each element e, we will denote by ϕ_α^e, $\alpha=1,\ldots,3$, the restrictions to e of the shape functions ϕ_i associated with the nodes of e.

2.4. Petrov–Galerkin FE approximation

The discrete Petrov–Galerkin FE problem expresses exactly as the Galerkin FE problem, Equation (4), but the test function space V^h has to be appropriately redefined by means of a set of global generating functions ψ_i, $i=1,\ldots,N$, whose local restrictions to element e are denoted by ψ_α^e, $\alpha=1,\ldots,3$. These test functions are first expressed as modifications of the Lagrange shape functions ϕ_α^e:

$$\psi_\alpha^e = \phi_\alpha^e + \tilde{\phi}_\alpha^e$$

with the perturbation functions $\tilde{\phi}_\alpha^e$ vanishing on the element boundaries to ensure discretization conformity. As an additional condition the linear space generated by the test functions is required to satisfy the partition of unity property (and hence to contain the constant functions) in order to keep the global and local conservation properties of the Galerkin method [14].

In the general problem, $\mathbf{v}$ is not constant but this case will be considered at a later stage and, in order to define the searched test functions, it will be first assumed that $\mathbf{v}$ is constant.

Our technique is based on searching bubble functions $\tilde{\phi}_\alpha^e$ that provide exact internal nodal values for a particular solution, denoted by $T^p(\mathbf{x})$, provided the exact nodal values of $T^p(\mathbf{x})$ be imposed along the domain boundary. In other words, we are not directly searching for a discrete solution to Equation (4), but a better discrete equation approximating Equation (1), which provides exact nodal values to the numerical solution in some particular cases. To this end, we consider a non-vanishing vector $\mathbf{m}$ and the following particular solution:

$$T^p(\mathbf{x}) = e^{(Pe(\mathbf{m}\cdot\mathbf{v})(\mathbf{m}\cdot\mathbf{x}))/\mathbf{m}\cdot\mathbf{m}}$$

This solution is sharp and badly resolved by the Galerkin method at high Pe. It will be briefly shown below that obtaining a nodally exact numerical approximation of $T^p(\mathbf{x})$ for any mesh from exact nodal boundary conditions only requires the following conditions to be satisfied by the 0th order moments (or average values) μ_α^e of the perturbations:

$$\begin{aligned}
\mu_1^e &= \frac{1}{S^e}\int_{S^e}\tilde{\phi}_1^e\,\mathrm{d}S = [(Pe_1^e - G_{11}^e)e^{-4G_{32}^e Pe_1^{e,\mathbf{m}}} - G_{12}^e e^{-4G_{13}^e Pe_2^{e,\mathbf{m}}} - G_{13}^e e^{-4G_{21}^e Pe_3^{e,\mathbf{m}}}]/D - \frac{1}{3}\\
\mu_2^e &= \frac{1}{S^e}\int_{S^e}\tilde{\phi}_2^e\,\mathrm{d}S = [(Pe_2^e - G_{22}^e)e^{-4G_{13}^e Pe_2^{e,\mathbf{m}}} - G_{23}^e e^{-4G_{21}^e Pe_3^{e,\mathbf{m}}} - G_{21}^e e^{-4G_{32}^e Pe_1^{e,\mathbf{m}}}]/D - \frac{1}{3}\\
\mu_3^e &= \frac{1}{S^e}\int_{S^e}\tilde{\phi}_3^e\,\mathrm{d}S = [(Pe_3^e - G_{33}^e)e^{-4G_{21}^e Pe_3^{e,\mathbf{m}}} - G_{31}^e e^{-4G_{32}^e Pe_1^{e,\mathbf{m}}} - G_{32}^e e^{-4G_{13}^e Pe_2^{e,\mathbf{m}}}]/D - \frac{1}{3}
\end{aligned} \tag{5}$$

where

$$D = Pe_1^e \mathrm{e}^{-4G_{32}^e Pe_1^{e,\mathbf{m}}} + Pe_2^e \mathrm{e}^{-4G_{13}^e Pe_2^{e,\mathbf{m}}} + Pe_3^e \mathrm{e}^{-4G_{21}^e Pe_3^{e,\mathbf{m}}} \tag{6}$$

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while S^e stands for both the domain and the area of element e, and $G^e_{\alpha\beta}$, Pe^e_α, and $Pe^{e,\mathbf{m}}_\alpha$ denote non-dimensional parameters. The coefficients $G^e_{\alpha\beta}$ are geometrical shape factors defined by

$$G^e_{\alpha\beta} = S^e \nabla\phi^e_\alpha \cdot \nabla\phi^e_\beta \tag{7}$$

while Pe^e_α and $Pe^{e,\mathbf{m}}_\alpha$ are element Péclet numbers given by

$$Pe^e_\alpha = Pe\, S^e \mathbf{v}\cdot\nabla\phi^e_\alpha, \quad Pe^{e,\mathbf{m}}_\alpha = \frac{Pe\, S^e (\mathbf{m}\cdot\mathbf{v})(\mathbf{m}\cdot\nabla\phi^e_\alpha)}{\mathbf{m}\cdot\mathbf{m}} \tag{8}$$

The demonstration of these conditions will be presented in detail in [15] and we only give here a sketch of the developments. To prove conditions (5), it is sufficient to observe that the equation provided by the test function ψ_i only involves the unknowns associated with the nodes that belong to the support $\mathrm{supp}(\phi_i)$ of ψ_i. This set of nodes will be denoted by $n(i)$; and hence, denoting $T^p(\mathbf{x}_n)$ by T_n, the equation associated with global node i is expressed as

$$\sum_{n\in n(i)} T_n \left[\int_{\mathrm{supp}(\phi_i)} (\phi_i + \tilde{\phi}_i)\mathbf{v}\cdot\nabla\phi_n \,\mathrm{d}\Omega + \frac{1}{Pe}\int_{\mathrm{supp}(\phi_i)} \nabla(\phi_i+\tilde{\phi}_i)\cdot\nabla\phi_n \,\mathrm{d}\Omega \right] = 0$$

Since the support of ψ_i consists of the set of elements $e(i)$ that share node i, each integral over $\mathrm{supp}(\phi_i)$ can be split into integrals over these different elements. For the sake of convenience, we here switch to a description involving a local node numbering and, without loss of generality, we assume that node i has 1 as local index for each element of $\mathrm{supp}(\phi_i)$ (these elements surround node i). Then, the equation is expressed as

$$\sum_{e\in e(i)} \sum_{\alpha=1}^{3} \frac{T^e_1}{T^e_\alpha} \left[\int_{\Omega^e} (\phi^e_1 + \tilde{\phi}^e_1)\mathbf{v}\cdot\nabla\phi^e_\alpha \,\mathrm{d}\Omega + \frac{1}{Pe}\int_{\Omega^e} \nabla(\phi^e_1+\tilde{\phi}^e_1)\cdot\nabla\phi^e_\alpha \,\mathrm{d}\Omega \right] = 0$$

Let us now observe that obviously for an internal mesh node i,

$$\sum_{e\in e(i)} Pe^e_i = 0 \quad \text{or} \quad \sum_{e\in e(i)} Pe^e_1 = 0 \text{ using the local node numbering}$$

Therefore, the equation for node i equivalently is rewritten as

$$\sum_{e\in e(i)} \sum_{\alpha=1}^{3} \frac{T^e_1}{T^e_\alpha} \left[\int_{\Omega^e} (\phi^e_1 + \tilde{\phi}^e_1)\mathbf{v}\cdot\nabla\phi^e_\alpha \,\mathrm{d}\Omega + \frac{1}{Pe}\int_{\Omega^e} \nabla(\phi^e_1+\tilde{\phi}^e_1)\cdot\nabla\phi^e_\alpha \,\mathrm{d}\Omega \right] + C \sum_{e\in e(i)} Pe^e_i = 0 \tag{9}$$

where C is an arbitrary constant for the entire domain. Integrating by parts the diffusive term on each element e readily shows that

$$\int_{\Omega^e} \nabla(\phi^e_\alpha + \tilde{\phi}^e_\alpha)\cdot\nabla\phi^e_\beta \,\mathrm{d}\Omega = \int_{\Omega^e} \nabla\phi^e_\alpha\cdot\nabla\phi^e_\beta \,\mathrm{d}\Omega$$

because $\tilde{\phi}^e_\alpha$ vanishes on the boundary of the element and ϕ^e_α is linear. Then the contribution E^e_i of element $e\in e(i)$ to the global equation i is expressed as

$$E^e_i = \sum_{\alpha=1}^{3} \frac{T^e_1}{T^e_\alpha} \left[\int_{\Omega^e} (\phi^e_1 + \tilde{\phi}^e_1)\mathbf{v}\cdot\nabla\phi^e_\alpha \,\mathrm{d}\Omega + \frac{1}{Pe}\int_{\Omega^e} \nabla(\phi^e_1)\cdot\nabla\phi^e_\alpha \,\mathrm{d}\Omega \right] + C Pe^e_1$$

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and the sum $\sum_{e\in e(i)} E_i^e$ on the elements $e(i)$ is constrained to vanish. Moreover, to let this sum vanish whatever the mesh, it is then sufficient to impose each of its terms to vanish, $E_i^e = 0$, while selecting $C = -1/Pe$ is required by the partition of unity constraint. Finally, the condition for μ_1^e is retrieved after some calculations and the other moment conditions, Equations (5), are obtained by applying a cyclic permutation of the indices $(1, 2, 3)$ to this condition.

It is easy to see that appropriate shape function perturbations can readily be determined from Equations (5) (hence providing the test functions ψ_α^e and the searched discrete formulation) in the case of a constant velocity **v** and direction **m**. Now the case of a variable velocity must be treated and the role of the direction vector **m** must be clarified. In the case of a non-constant **v**, two simple procedures can be considered either by defining an average velocity on each element e from which the perturbation functions $\tilde{\phi}_\alpha^e$ are directly obtained by Equations (5)–(8) or by averaging the different sets of perturbation functions provided by the different nodal velocities of each element. Both procedures provide continuous test functions with respect to the partition of unity property. On the other hand, the test functions just found depend on **m**. Any technique able to locally approximate the solution by a 1D expression of the form $T^p(\mathbf{x})$ should provide a convenient **m**, which hence becomes a problem unknown assumed to be constant on each element and from which the test functions are obtained by Equations (5)–(8). The general problem, which resumes searching for the most appropriate direction vector **m** associated with a variable **v** and possibly to the domain shape, lies beyond the scope of the present analysis and further results will be published in subsequent papers.

3. RESULTS

Some typical results [5] obtained by means of the CPG method are here compared with their counterparts obtained from the classical SUPG method [2].

3.1. Two-boundary layer problem

The problem depicted in Figure 1 takes its interest from the fact that the exact solution presents two sharp boundary layers on the outflow boundaries at high *Pe*.

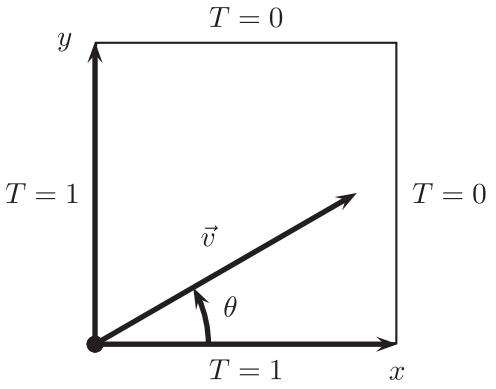

Figure 1. Two-boundary layer problem.

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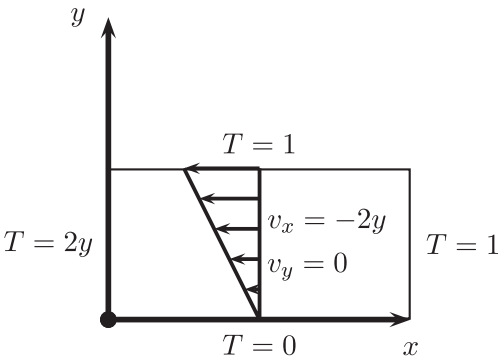

Figure 2. Thermal boundary layer problem.

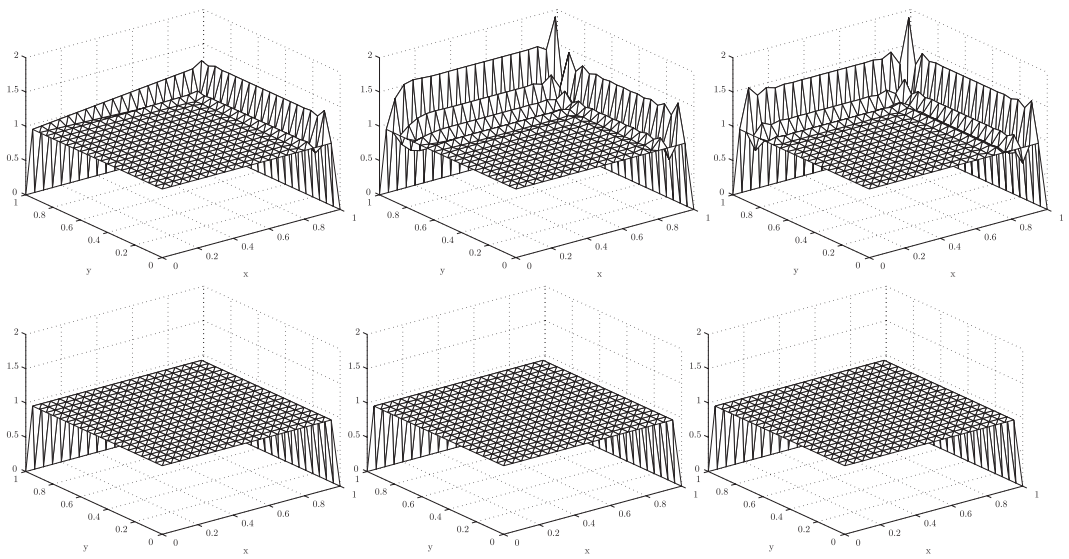

Figure 3. Two-boundary layer problem for $\theta = 1, 30$ and 45° (from left to right). CPG results (bottom) are far better than SUPG results (top).

Numerical experiments are performed for $Pe = 10^6$ on a 20×20 mesh made of rectangular triangles. The solution obtained by using the SUPG method exhibits strong overshoots while the CPG method provides an almost nodally exact solution for any orientation θ of the velocity vector (Figure 3). For the CPG result, **m** was taken as normal to the outflow boundary for the elements having node(s) on that boundary while it was taken as equal to **v** for the other elements.

3.2. Thermal boundary layer problem

The problem is depicted in Figure 2. This situation may be viewed as modelling the formation of a pair of thermal boundary layers along the lower and outflow boundaries of a fully developed

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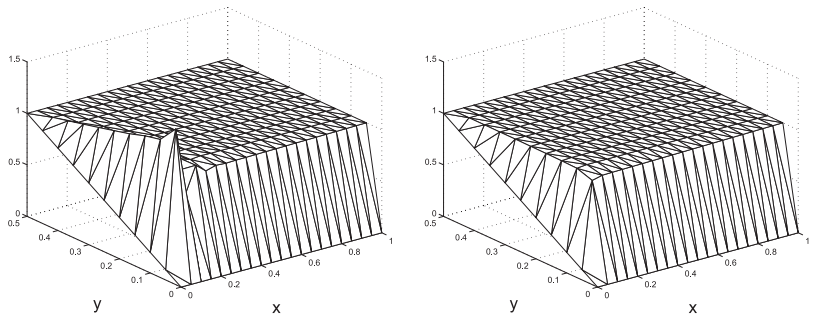

Figure 4. Thermal boundary layer problem: SUPG (left) and CPG (right) results.

shear flow between two parallel plates, where the upper plate moves to the left while the bottom plate is fixed.

Numerical experiments are performed with $Pe = 10^5$ on a 20×10 mesh made of rectangular triangles. Figure 4 compares the results obtained with the SUPG method and the CPG method (with $\mathbf{m} = \mathbf{v}$). We observe that the solution provided by the CPG method clearly behaves much better than the SUPG solution. In particular, it is noticeable that the SUPG method generates in the downstream boundary layer a strong overshoot that propagates upstream in the lower boundary layer, while the CPG method behaves much better.

4. CONCLUSIONS

A truly CPG FE method to solve convection-dominated problems has been proposed. Our approach is based on using test functions that provide exact nodal values for a selected class of solutions. Results of very high quality have been obtained. In order to build a general discretization and solution algorithm, the research effort will be pursued, focusing on the selection of the numerical direction $\mathbf{m}$ according to an appropriate criterion and the use of higher order elements (it can be mentioned that our numerical experiments already show that a good $\mathbf{m}$ should always be normal to the system boundary layers). Additional investigations will be devoted to handle more general classes of boundary conditions and to determine an appropriate weighting factor between the pure Galerkin and CPG methods in order to avoid the occurrence of unbounded test functions at very low Pe.

It is worth nothing that this research clearly indicates that accurate, reliable and CPG FE methods can be built. In addition, the method we propose directly leads to defining appropriate dimensionless element Péclet numbers and geometrical factors in order to exactly characterize the local transport intensity and orientation with respect to the element shape, hence providing rigorous tools to solve the convection–diffusion problem on general unstructured meshes.

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